THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

VOLUME FOUR OF FOUR

SECTION 24

RELATED PROBLEMS 41 THRU 48

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PROBLEM NUMBER 41 *(Ref. Section 14.2)*

GIVEN:

The following *Generalized Cubic Function*:

$$z^3 + 30z^2 + 300z + (30 - 299\zeta) = y$$

Where,

$$\zeta_{TOPSURFACE} = \tan 255^o$$

$$\zeta_{BOTSURFACE} = \tan 15^o$$

DETERMINE:

How to generate a *two-dimensional surface* which is bordered by the given *top and bottom curves*

SOLUTION:

The given *Generalized Cubic Equation* characterizes a *family* of curves which are contained within the given *top and bottom curves*.

For each and every *z-coordinate specified*, any two curves individually selected from this *family* differ by an ordinate value $299(\zeta_2 - \zeta_1)$, as determined below:

Where,

$$z_2^3 + 30z_2^2 + 300z_2 + (30 - 299\zeta_2) = y_2$$

However, when $z_2 = z_1$,

$$z_1^3 + 30z_1^2 + 300z_1 + (30 - 299\zeta_2) = y_2$$

$$z_1^3 + 30z_1^2 + 300z_1 = y_2 - (30 - 299\zeta_2)$$

But,

$$z_1^3 + 30z_1^2 + 300z_1 + (30 - 299\zeta_1) = y_1$$

Substitution into this lower function confirms the above premise as follows:

$$(z_1^3 + 30z_1^2 + 300z_1) + (30 - 299\zeta_1) = y_1$$

$$y_2 - (30 - 299\zeta_2) + (30 - 299\zeta_1) = y_1$$

$$y_2 - 30 + 299\zeta_2 + 30 - 299\zeta_1 = y_1$$

$$y_2 + 299(\zeta_2 - \zeta_1) = y_1$$

$$299(\zeta_2 - \zeta_1) = y_1 - y_2$$

Accordingly, a *two-dimensional surface* may be generated by plotting such various *family* functions where each is to be altered by a small variation in the value of ζ, say tan 5°.

As it may become necessary to introduce more *family* functions in order to complete the detailed surface, this tan 5° variation may be reduced proportionately.

The plot of this *two-dimensional surface* appears in *Figure 75* where abscissa values range between ±15 units. Respective y-plot determinations appear in *Table 45.*

Figure 75. Two-dimensional Surface Generation..

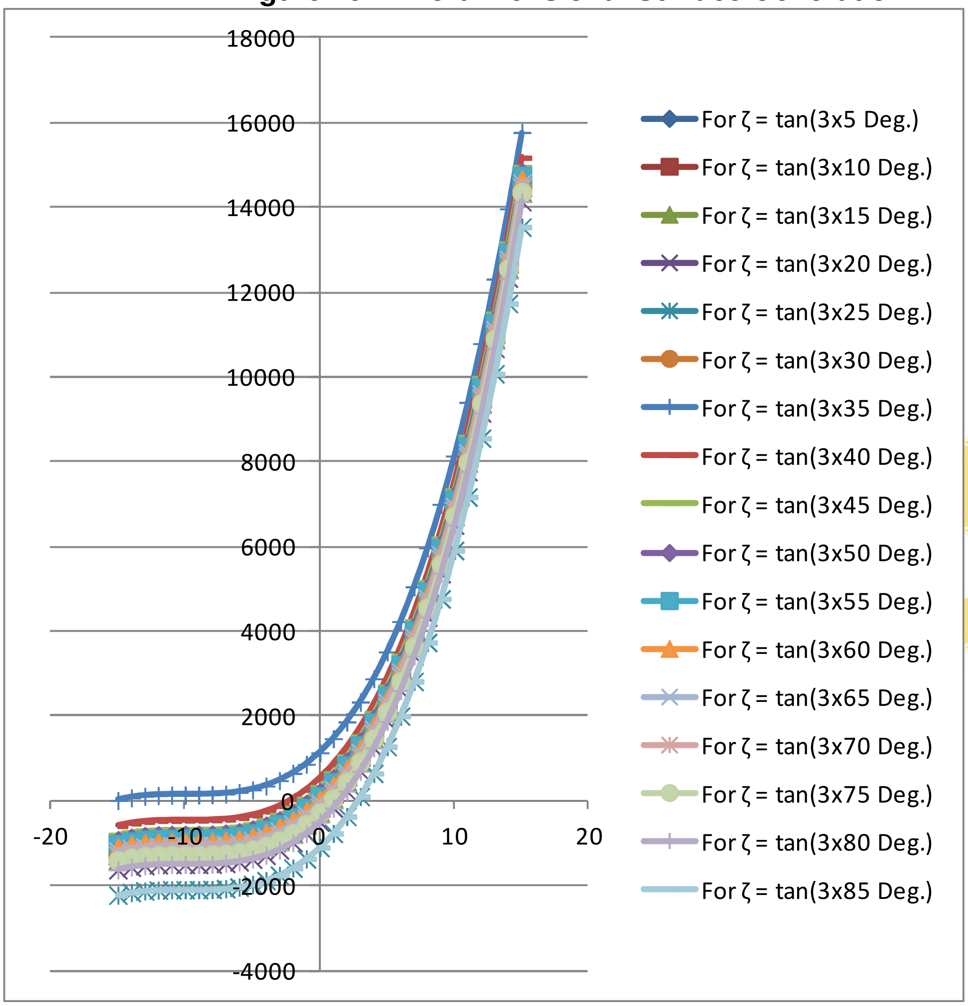


Table 45. Two-dimensional Surface Generation Plot Points.

z	y plot values								
	For ζ = tan15°	For ζ = tan30°	For ζ = tan 45°	For ζ = tan60°	For ζ = tan 75°	For ζ = tan 90°	For ζ = tan 105°	For ζ = tan 120°	For ζ = tan 135°
15	14574.88319	14482.37227	14356	14137.11681	13539.11681	14655	15770.88319	15172.88319	14954
14	12773.88319	12681.37227	12555	12336.11681	11738.11681	12854	13969.88319	13371.88319	13153
13	11116.88319	11024.37227	10898	10679.11681	10081.11681	11197	12312.88319	11714.88319	11496
12	9597.883192	9505.37227	9379	9160.116808	8562.116808	9678	10793.88319	10195.88319	9977
11	8210.883192	8118.37227	7992	7773.116808	7175.116808	8291	9406.883192	8808.883192	8590
10	6949.883192	6857.37227	6731	6512.116808	5914.116808	7030	8145.883192	7547.883192	7329
9	5808.883192	5716.37227	5590	5371.116808	4773.116808	5889	7004.883192	6406.883192	6188
8	4781.883192	4689.37227	4563	4344.116808	3746.116808	4862	5977.883192	5379.883192	5161
7	3862.883192	3770.37227	3644	3425.116808	2827.116808	3943	5058.883192	4460.883192	4242
6	3045.883192	2953.37227	2827	2608.116808	2010.116808	3126	4241.883192	3643.883192	3425
5	2324.883192	2232.37227	2106	1887.116808	1289.116808	2405	3520.883192	2922.883192	2704
4	1693.883192	1601.37227	1475	1256.116808	658.1168084	1774	2889.883192	2291.883192	2073
3	1146.883192	1054.37227	928	709.1168084	111.1168084	1227	2342.883192	1744.883192	1526
2	677.8831922	585.3722696	459	240.1168084	-357.8831916	758	1873.883192	1275.883192	1057
1	280.8831922	188.3722696	62	-156.8831916	-754.8831916	361	1476.883192	878.8831916	660
0	-50.11680781	-142.6277304	-269	-487.8831916	-1085.883192	30	1145.883192	547.8831916	329
-1	-321.1168078	-413.6277304	-540	-758.8831916	-1356.883192	-241	874.8831916	276.8831916	58
-2	-538.1168078	-630.6277304	-757	-975.8831916	-1573.883192	-458	657.8831916	59.88319159	-159
-3	-707.1168078	-799.6277304	-926	-1144.883192	-1742.883192	-627	488.8831916	-109.1168084	-328
-4	-834.1168078	-926.6277304	-1053	-1271.883192	-1869.883192	-754	361.8831916	-236.1168084	-455
-5	-925.1168078	-1017.62773	-1144	-1362.883192	-1960.883192	-845	270.8831916	-327.1168084	-546
-6	-986.1168078	-1078.62773	-1205	-1423.883192	-2021.883192	-906	209.8831916	-388.1168084	-607
-7	-1023.116808	-1115.62773	-1242	-1460.883192	-2058.883192	-943	172.8831916	-425.1168084	-644
-8	-1042.116808	-1134.62773	-1261	-1479.883192	-2077.883192	-962	153.8831916	-444.1168084	-663
-9	-1049.116808	-1141.62773	-1268	-1486.883192	-2084.883192	-969	146.8831916	-451.1168084	-670

z	y plot values								
	For ζ = tan15°	For ζ = tan30°	For ζ = tan 45°	For ζ = tan60°	For ζ = tan 75°	For ζ = tan 90°	For ζ = tan 105°	For ζ = tan 120°	For ζ = tan 135°
-10	-1050.116808	-1142.62773	-1269	-1487.883192	-2085.883192	-970	145.8831916	-452.1168084	-671
-11	-1051.116808	-1143.62773	-1270	-1488.883192	-2086.883192	-971	144.8831916	-453.1168084	-672
-12	-1058.116808	-1150.62773	-1277	-1495.883192	-2093.883192	-978	137.8831916	-460.1168084	-679
-13	-1077.116808	-1169.62773	-1296	-1514.883192	-2112.883192	-997	118.8831916	-479.1168084	-698
-14	-1114.116808	-1206.62773	-1333	-1551.883192	-2149.883192	-1034	81.88319159	-516.1168084	-735
-15	-1175.116808	-1267.62773	-1394	-1612.883192	-2210.883192	-1095	20.88319159	-577.1168084	-796

z	Y plot values							
	For ζ = tan150°	For ζ = tan 165°	For ζ = tan 180°	For ζ = tan195°	For ζ = tan210°	For ζ = tan225°	For ζ = tan 240°	For ζ = tan 255°
15	14827.71743	14735.11681	14655	14574.88319	14482.3865	14356	14137.11681	13539.11681
14	13026.71743	12934.11681	12854	12773.88319	12681.3865	12555	12336.11681	11738.11681
13	11369.71743	11277.11681	11197	11116.88319	11024.3865	10898	10679.11681	10081.11681
12	9850.71743	9758.116808	9678	9597.883192	9505.386496	9379	9160.116808	8562.116808
11	8463.71743	8371.116808	8291	8210.883192	8118.386496	7992	7773.116808	7175.116808
10	7202.71743	7110.116808	7030	6949.883192	6857.386496	6731	6512.116808	5914.116808
9	6061.71743	5969.116808	5889	5808.883192	5716.386496	5590	5371.116808	4773.116808
8	5034.71743	4942.116808	4862	4781.883192	4689.386496	4563	4344.116808	3746.116808
7	4115.71743	4023.116808	3943	3862.883192	3770.386496	3644	3425.116808	2827.116808
6	3298.71743	3206.116808	3126	3045.883192	2953.386496	2827	2608.116808	2010.116808
5	2577.71743	2485.116808	2405	2324.883192	2232.386496	2106	1887.116808	1289.116808
4	1946.71743	1854.116808	1774	1693.883192	1601.386496	1475	1256.116808	658.1168084
3	1399.71743	1307.116808	1227	1146.883192	1054.386496	928	709.1168084	111.1168084
2	930.7174304	838.1168084	758	677.8831916	585.3864957	459	240.1168084	-357.8831916
1	533.7174304	441.1168084	361	280.8831916	188.3864957	62	-156.8831916	-754.8831916
0	202.7174304	110.1168084	30	-50.11680841	-142.6135043	-269	-487.8831916	-1085.883192
-1	-68.28256957	-160.8831916	-241	-321.1168084	-413.6135043	-540	-758.8831916	-1356.883192
-2	-285.2825696	-377.8831916	-458	-538.1168084	-630.6135043	-757	-975.8831916	-1573.883192
-3	-454.2825696	-546.8831916	-627	-707.1168084	-799.6135043	-926	-1144.883192	-1742.883192
-4	-581.2825696	-673.8831916	-754	-834.1168084	-926.6135043	-1053	-1271.883192	-1869.883192
-5	-672.2825696	-764.8831916	-845	-925.1168084	-1017.613504	-1144	-1362.883192	-1960.883192
-6	-733.2825696	-825.8831916	-906	-986.1168084	-1078.613504	-1205	-1423.883192	-2021.883192
-7	-770.2825696	-862.8831916	-943	-1023.116808	-1115.613504	-1242	-1460.883192	-2058.883192
-8	-789.2825696	-881.8831916	-962	-1042.116808	-1134.613504	-1261	-1479.883192	-2077.883192

z	Y plot values							
	For ζ = tan150°	For ζ = tan 165°	For ζ = tan 180°	For ζ = tan195°	For ζ = tan210°	For ζ = tan225°	For ζ = tan 240°	For ζ = tan 255°
-9	-796.2825696	-888.8831916	-969	-1049.116808	-1141.613504	-1268	-1486.883192	-2084.883192
-10	-797.2825696	-889.8831916	-970	-1050.116808	-1142.613504	-1269	-1487.883192	-2085.883192
-11	-798.2825696	-890.8831916	-971	-1051.116808	-1143.613504	-1270	-1488.883192	-2086.883192
-12	-805.2825696	-897.8831916	-978	-1058.116808	-1150.613504	-1277	-1495.883192	-2093.883192
-13	-824.2825696	-916.8831916	-997	-1077.116808	-1169.613504	-1296	-1514.883192	-2112.883192
-14	-861.2825696	-953.8831916	-1034	-1114.116808	-1206.613504	-1333	-1551.883192	-2149.883192
-15	-922.2825696	-1014.883192	-1095	-1175.116808	-1267.613504	-1394	-1612.883192	-2210.883192

PROBLEM NUMBER 42 *(Ref. Section 14.2)*

GIVEN:

The *Three Theta Cubic Function* rendered below for $3\phi = 60^o$:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y$$

GRAPHICALLY DEPICT:

a) The relationship of the *given Function* to the *so-called Perfect Cubic Function* and a *Straight Line:*

$$z^3 = y \qquad \textit{(Ref. Equation 46)}$$

b) The relationship to the *so-called Trigonometric Solution of the Cubic Equation*[1]

SOLUTION:

a) The objective here is to transform the given Function into a *Generalized Cubic Function devoid* of its *second term*. This produces a function of the following form which is a combination of a straight line and the *so-called Perfect Cubic Function* denoted above.

$$z'^3 + \tau z' + \nu = y'$$

This is accomplished by *shifting* the *initial origin* to the right by a distance of β′/3 calculated using the *Generalized Cubic Equation* as follows:

Where:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \frac{\beta}{\alpha} z^2 + \frac{\gamma}{\alpha} z + \frac{\delta}{\alpha} = 0$$

$$z^3 + \beta' z^2 + \gamma' z + \delta' = 0$$

This equation now is perceived from a *relocated origin (Ref. Section 14.2)* by letting $z' = z - f$ as follows:

$$z^3 + \beta' z^2 + \gamma' z + \delta' = 0$$

$$(z' + f)^3 + \beta'(z' + f)^2 + \gamma'(z' + f) + \delta' = 0$$

$$z'^3 + 3fz'^2 + 3f^2 z' + f^3 + \beta'(z'^2 + 2fz' + f^2) + \gamma'(z' + f) + \delta' = 0$$

$$z'^3 + (3f + \beta')z'^2 + (3f^2 + 2f\beta' + \gamma')z' + (f^3 + \beta' f^2 + \gamma' f + \delta') = 0$$

The *coefficient* of the 2nd term is set to equal zero:

$$3f + \beta' = 0$$

$$f = -\frac{\beta'}{3}$$

[1] CRC Standard Mathematical Tables 23rd Edition; Samuel M. Selby - Editor in Chief; CRC Press, Inc, Cleveland Ohio; 1975; page 104.

Then,

$$\sigma = 3f + \beta'$$
$$= 0$$

$$\tau = 3f^2 + 2f\beta' + \gamma'$$
$$= \frac{\beta'^2}{3} - \frac{2\beta'^2}{3} + \gamma'$$
$$= \gamma' - \frac{\beta'^2}{3}$$

$$\nu = f^3 + \beta' f^2 + \gamma' f + \delta'$$
$$= -\frac{\beta'^3}{27} + \frac{\beta'^3}{9}(\frac{3}{3}) - \frac{\beta'\gamma'}{3}(\frac{9}{9}) + \frac{27\delta'}{27}$$
$$= \frac{1}{27}(2\beta'^3 - 9\beta'\gamma + 27\delta')$$

For the *Theta Cubic Function*:

$$\beta' = -3\zeta$$
$$\gamma' = -3$$
$$\delta' = \zeta =$$

Then,

$$z'^3 + \tau z' + \nu = y'$$
$$z'^3 + (\gamma' - \frac{\beta'^2}{3})z' + \frac{1}{27}(2\beta'^3 - 9\beta'\gamma + 27\delta') =$$
$$z'^3 - 3(1+\zeta^2)z' + \frac{1}{27}[-2(27)\zeta^3 - 3(27)\zeta + 27\zeta] =$$
$$z'^3 - 3(1+\zeta^2)z' - 2\zeta^3 - 3\zeta + \zeta =$$
$$z'^3 - 3(1+\zeta^2)z' - 2\zeta(1+\zeta^2) =$$

When $3\phi = 60^o$,

$$\zeta = \tan(3\phi) = \tan 60^o = \sqrt{3}$$

Substitution produces:

$$z'^3 - 3(1+3)z' - 2\sqrt{3}(1+3) = y'$$
$$z'^3 - 12z' - 8\sqrt{3} = y'$$

The roots, z_R', z_S', and z_T' occur as $y' = 0$ as follows:

$$z'^3 - 12z' - 8\sqrt{3} = 0$$

Or,

$$z'^3 = 12z' + 8\sqrt{3}$$

The three following functions then are plotted upon the same grid *(Ref. Figure 76)*:

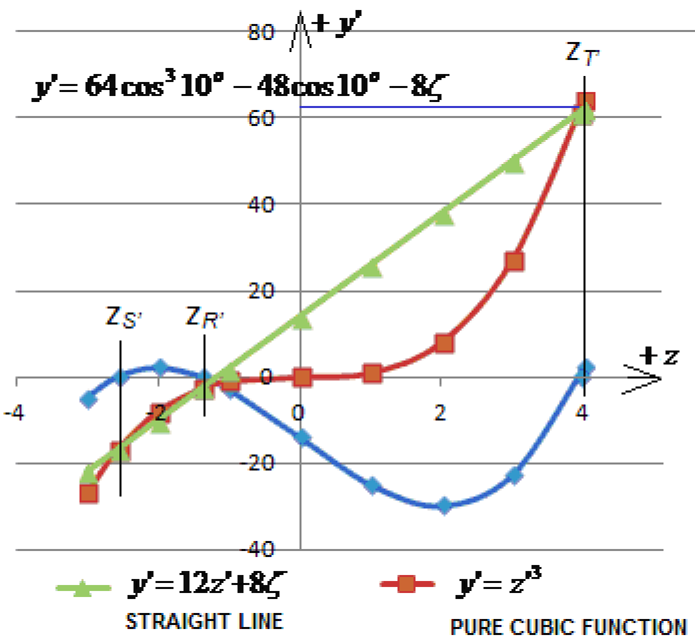
$$12z'+8\sqrt{3}-y'$$

$$z'^3=y'$$

$$z'^3-12z'-8\sqrt{3}=y'$$

As indicated, *intersection points* between the first two functions listed directly above occur at respective vertical projections of the three roots belonging to the third function.

Figure 76. *$z'^3+\tau\, z'^3+v$ as a Perfect Cube Function and a Straight Line.*



z'	$z'^3-12z'-8\sqrt{3}=y$	$z'^3=y'$	$12z'+8\sqrt{3}-y'$
5	51.14359354	125	73.85640646
4	2.143593539	64	61.85640646
3.939231012	0	61.1271786	61.1271786
3	-22.85640646	27	49.85640646
2	-29.85640646	8	37.85640646
1	-24.85640646	1	25.85640646
0	-13.85640646	0	13.85640646
-1	-2.856406461	-1	1.856406461
-1.368080573	0	-2.560560417	-2.560560415
-2	2.143593539	-8	-10.14359354
-2.571150439	0	-16.99739881	-16.99739881
-3	-4.856406461	-27	-22.14359354
-4	-29.85640646	-64	-34.14359354
-5	-78.85640646	-125	-46.14359354

b) The *above referenced so-called Trigonometric Solution of the Cubic Equation more realistically* depicts an actual ***proportional*** than *Trigonometric Solution*.

As a *misnomer*, or *dangling participle*, such algorithm doesn't employ a *trigonometric approach*, simply because no *Euclidean construction*, via straight edge and compass alone, can be applied in order to resolve it!

Instead, a more *appropriate title* shall hereinafter be referred to as a *Proportional Solution of the Cubic Equation* as *evidenced* in the analysis presented below.

For the above equation:

$$z'^3 - 12z' - 8\sqrt{3} = 0$$

Letting $z' = m\cos\theta$ yields *(Ref. Section 13.3.6)*:

$$m^3\cos^3\theta - 12m\cos\theta - 8\sqrt{3} = 0$$

$$\cos^3\theta - \frac{12}{m^2}\cos\theta - \frac{8\sqrt{3}}{m^3} = 0$$

Where,

$$4\cos^3\theta - 3\cos\theta = \cos(3\theta) \quad \textit{(Ref. Equation 1)}$$

$$\cos^3\theta \quad \frac{3}{4}\cos\theta \quad \frac{\cos(3\theta)}{4} = 0$$

Comparing respective coefficients of like terms renders the following ***proportions***:

$$-\frac{3}{4} = -\frac{12}{m^2}$$

$$--\frac{\cos(3\theta)}{4} = -\frac{8\sqrt{3}}{m^3}$$

Then,

$$m^2 = 12(\frac{4}{3})$$

$$= 4^2$$

Or,

$$m = \pm 4$$

$$-\frac{\cos(3\theta)}{4} = -\frac{8\sqrt{3}}{m^3}$$

$$= \mp\frac{8\sqrt{3}}{4^3}$$

Or,

$$\cos(3\theta) = \pm\frac{\sqrt{3}}{2}$$
$$3\theta = \pm 30^o$$
$$\theta = \pm 10^o$$

Therefore,

$$z' = m\cos\theta$$
$$= \pm 4(\pm\cos 10^o)$$
$$= +4\cos 10^o$$
$$= +3.939231012$$

Check,

$$z' + \zeta = z$$
$$+3.939231012 + \sqrt{3} =$$
$$5.67128182 =$$
$$\tan 80^o =$$
$$\tan(4\phi) = z_T \qquad \textit{Q.E.D.}$$

And,

$$z'^3 - 12z' - 8\sqrt{3} = y$$
$$(4\cos 10^o)^3 - 12(4\cos 10^o) - 8\sqrt{3} =$$
$$64(\cos 10^o)^3 - 48\cos 10^o - 8\zeta =$$

PROBLEM NUMBER 43 *(Ref. Section 16)*

GIVEN:

The *Perfect Cubic Function (Ref. Equation 46)*:

$$y = z^3$$

GRAPHICALLY DEPICT:

The associated *curve family* for the following *curve shape:*

$$y' = z'^3 + 6z'^2 + 12z' + 8$$

SOLUTION:

The *given function* is first *displaced horizontally* without changing its *curve shape*. This is accomplished simply by referring to the *given stationary function* from a *relocated origin (Ref. Section 14.2)* by letting $z' = z - f$ as follows:

$$y = z^3$$
$$= (z' + f)^3$$
$$= z'^3 + 3fz'^2 + 3f^2z' + f^3$$

Since the *above determined transform* is only displaced horizontally:

$$y = z'^3 + 3fz'^2 + 3f^2z' + f^3 = y'$$

Comparing respective coefficients of like terms of the curve shape $y' = z'^3 + 6z'^2 + 4z' + 8$ renders the following *proportions*:

$$\beta' = 3f = +6$$

$$\gamma' = 3f^2 = 12$$

$$\delta' = f^3 = 8$$

For all cases:

$$f = +2$$

Hence, the curve shape $y' = z'^3 + 6z'^2 + 4z' + 8$ is identical to the *given Perfect Cubic Function*. The associated *parent curve* is of the following form:

$$z''^3 + \sigma z''^2 + \nu = y'' = y' = y$$

Where,

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$\nu = \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$

Such that:

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$
$$= -\sqrt{6^2 - 3(12)}$$
$$= 0$$

$$\nu = \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$
$$= \frac{1}{27}[2\beta'^3 - 2(3\gamma' - \beta'^2)\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$
$$= \frac{1}{27}[2\beta'^3 + 2(-\sigma^2)\sigma - 9\beta'\gamma' + 27\delta']$$
$$= \frac{1}{27}[2\beta'^3 - 2(\sigma^2)\sigma - 9\beta'\gamma' + 27\delta']$$
$$= \frac{1}{27}[2(6)^3 - 2(0) - 9(6)(12) + 27(8)]$$
$$= \frac{1}{27}[12(36) - 54(12 + 4)]$$
$$= \frac{1}{9(3)}[12(9)(4) - 9(6)(16)]$$
$$= 16 - 32$$
$$= -16$$

Hence the *parent curve* is:

$$z''^3 + \sigma z''^2 + \nu = y$$
$$z''^3 - 32 = y$$

The following curves are shown in *Figure* 77 plotted besides one another:

- A *typical family curve*, otherwise determined as the *displaced Perfect Cubic Function*

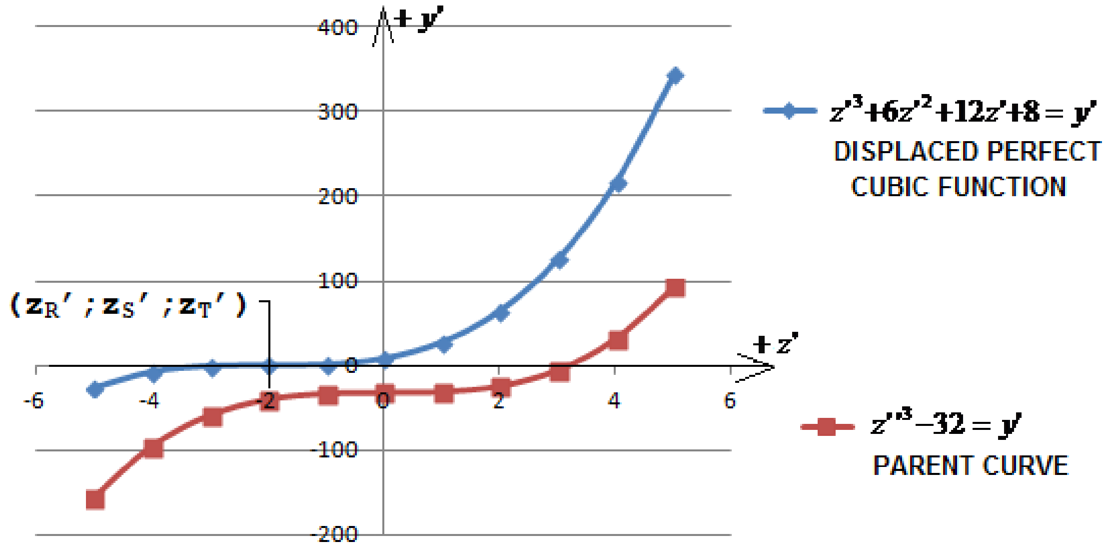
  $$z'^3 + 6z'^2 + 12z' + 8 = y'$$

- The *Parent Curve*

  $$z''^3 - 32 = y'$$

Notice that the *Parent Curve* displays a *common relative low and high point (Ref. Section 14.2.2)* upon the y′-axis at a distance of ν below the origin, whereas the displaced *Perfect Cubic Function* illustrates its *common relative low and high point* upon the z′-axis at the mutual location where the *three roots* occur.

Figure 77. ***Perfect Cube Function*** **Family Curves.**



z'	$z'^3+6z'^2+12z'+8=y'$	$z'^3-32=y'$
5	343	93
4	216	32
3	125	-5
2	64	-24
1	27	-31
0	8	-32
-1	1	-33
-2	0	-40
-3	-1	-59
-4	-8	-96
-5	-27	-157

PROBLEM NUMBER 44 *(Ref. Section 18.5)*

GIVEN:

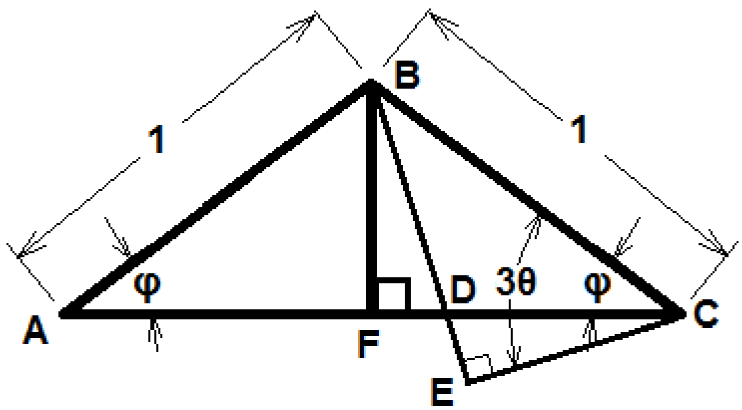
The isosceles triangle ABC upon which right triangle BCE is superimposed *(Ref. Figure 78)*, such that $\angle BCE = 3\theta$;

Where,

$$\tau = \cos(3\theta) = \overline{CE}$$

$$\eta = \sin(3\theta) = \overline{BE}$$

Figure 78. Isosceles Triangle ABC along with 3θ Right Triangle BCE.



DEMONSTRATE:

a) $\varphi = 2\theta$ when $\overline{AD} = 1$

b) $\overline{CD}\sin(2\theta) = \tau(\eta - \overline{CD}\sin\theta)$

c) $\overline{CD}\sin(2\theta) = \tau[\eta - \overline{CD}\sin\theta]$ cannot by ***determined*** via *Euclidean construction*

SOLUTION:

a) Since the interior angles of any triangle total 180°:

For right triangle BCE,

$$\angle BCE + 90^o + \angle CBE = 180^o$$

$$\angle BCE + \angle CBE = 90^o$$

$$\angle CBE = 90^o - \angle BCE$$

$$= 90^o - 3\theta$$

When $\overline{AD} = 1$, triangle ABD becomes isosceles, such that:

$$\angle ABD + \angle BDA + \angle DAB = 180^o$$

$$\angle ABD + \angle ABD + \varphi = 180^o$$

$$\angle ABD = \frac{180^o - \varphi}{2} = \angle BDA$$

Furthermore,

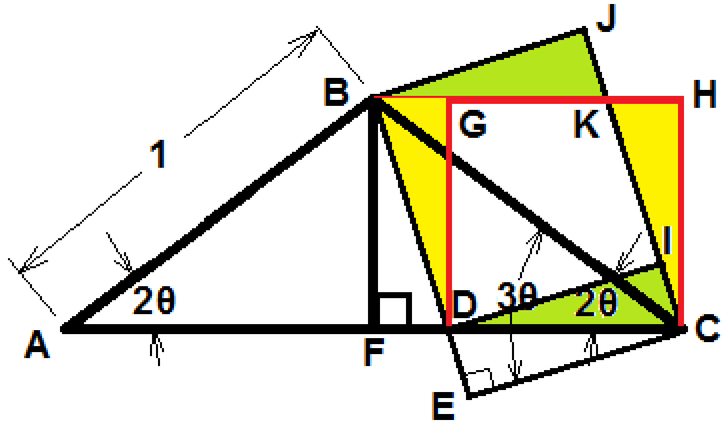
$$\angle ABC = \angle ABD + \angle CBE$$
$$= \frac{180^o - \varphi}{2} + (90^o - 3\theta)$$
$$= \frac{360^o - \varphi - 6\theta}{2}$$

Then, for isosceles triangle ABC,

$$\angle ABC + \angle BCA + \angle CAB = 180^o$$
$$\frac{360^o - \varphi - 6\theta}{2} + \varphi + \varphi = 180^o$$
$$\frac{360^o - \varphi - 6\theta + 4\varphi}{2} = \frac{360^o}{2}$$
$$3\varphi - 6\theta = 0$$
$$3\varphi = 6\theta$$
$$\varphi = 2\theta$$

b) *Figure 79* depicts isosceles triangle ABC along with 3θ right triangle BCE when $\varphi = 2\theta$ *(Ref. above Solution)*.

Figure 79. ***Figue 78*** **Construction for φ=2θ.**



When lines $\overline{DG}$ and $\overline{CH}$ are drawn parallel to line $\overline{BF}$ such that straight line $\overline{BGH}$ is constructed parallel to line $\overline{AC}$,

$$\overline{AB}\sin(2\theta) = \overline{BF} = \overline{DG} = \overline{CH} = 1\sin(2\theta) = \sin(2\theta)$$

Then,

$$A_{CDGH} = \overline{CD}x\overline{DG} = \overline{CD}\sin(2\theta)$$

Secondly, when lines $\overline{DI}$ and $\overline{BJ}$ are drawn parallel to line $\overline{EC}$ such that straight line $\overline{CIJ}$ is constructed parallel to line $\overline{BE}$,

Then,

$$
\begin{aligned}
A_{BDIJ} &= \overline{DI}x\overline{BD} \\
&= \overline{CE}(\overline{BE} - \overline{DE}) \\
&= \overline{CE}[\eta - \overline{CE}\tan(3\theta - 2\theta)] \\
&= \overline{CE}[\eta - \overline{CE}\tan\theta) \\
&= \overline{CE}(\eta - \tau\frac{\sin\theta}{\cos\theta}) \\
&= \overline{CE}(\frac{\eta\cos\theta - \tau\sin\theta}{\cos\theta}) \\
&= \overline{\frac{CE}{\cos\theta}[\sin(3\theta - \theta)]} \\
&= \overline{CD}\sin(2\theta) \\
&= A_{CDGH}
\end{aligned}
$$

Both areas expressed above are equal because they are comprised solely of sections of the same exact size as follows:

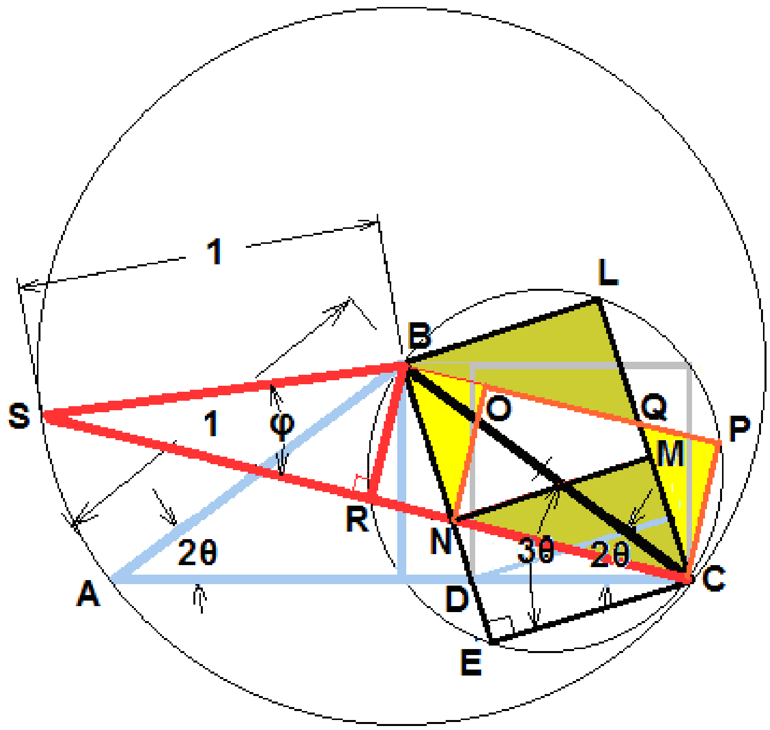
- A_{GDIK} is common to both *areas CDGH* and *BDIJ*
- A_{CHK} is common to *area CDGH* and is equal to A_{BDG} which is common to *area BDIJ*
- A_{CDI} is common to *area CDGH* and is equal to A_{BKJ} which is common to *area BDIJ*

Therefore,

$$
\begin{aligned}
A_{CDGH} &= A_{BDIJ} \\
\overline{CD}x\overline{DG} &= \overline{DI}x\overline{BD} \\
\overline{CD}x\overline{BF} &= \overline{CE}x(\overline{BE} - \overline{DE}) \\
\overline{CD}\sin(2\theta) &= \tau[\eta - \overline{CD}\sin(3\theta - 2\theta)] \\
\overline{CD}\sin(2\theta) &= \tau(\eta - \overline{CD}\sin\theta)
\end{aligned}
$$

c) *Figure 80* depicts isosceles triangle SBC in relationship to triangle ABC, along with 3θ right triangle BCE. In triangle SBC, $\angle BSC = \angle BCS = \varphi$

Figure 80. ***Figure 78*** **Construction for φ≠2θ.**



Such that,

$$A_{CNOP} = A_{BLMN}$$

The above equation holds because both areas expressed are comprised solely of sections of the same exact size as follows:

- A_{MNOQ} is common to both *areas CNOP* and *BLMN*
- A_{CPQ} is common to *area CNOP* and is equal to A_{BNO} which is common to *area BLMN*
- A_{CMN} is common to *area CNOP* and is equal to A_{BLQ} which is common to *area BLMN*

Where,

$$\overline{MN} = \overline{CE} = \tau$$

$$\overline{BN} = \overline{BE} - \overline{NE}$$
$$= \eta - \overline{CN}\sin(3\theta - \varphi)$$

$$\overline{BS}\sin(\varphi) = \overline{BR} = \overline{ON} = \overline{CP} = 1\sin\varphi = \sin\varphi$$

Then,

$$A_{CNOP} = A_{BLMN}$$

Or,

$$\overline{CN} x \overline{CP} = \overline{MN} x \overline{BN}$$
$$\overline{CN} x \overline{CP} = \overline{CE}(\overline{BE} - \overline{NE})$$
$$\overline{CN}\sin\varphi = \tau[\eta - \overline{CN}\sin(3\theta - \varphi)]$$

Since the line $\overline{BC}$ is common to both triangle ABC and to triangle BCS, the two become identical <u>only</u> when $\varphi = 2\theta$. When this occurs, line $\overline{CN}$ becomes synonymous with line $\overline{CD}$ such that:

$$\overline{CN}\sin\varphi = \tau[\eta - \overline{CN}\sin(3\theta - \varphi)]$$
$$\overline{CD}\sin(2\theta) = \tau(\eta - \overline{CD}\sin\theta)$$

However, the overall equation $\overline{CN}\sin\varphi = \tau[\eta - \overline{CN}\sin(3\theta - \varphi)]$ encompasses *other* solutions which are different than the one noted above.

With *multiple solutions*, the *Euclidean application* does not know which one to construct. Hence, it <u>cannot</u> differentiate $\overline{CD}\sin(2\theta) = \tau(\eta - \overline{CD}\sin\theta)$ from other constructions which can be realized from the equation $\overline{CN}\sin\varphi = \tau[\eta - \overline{CN}\sin(3\theta - \varphi)]$.

Even when $\overline{AD} = 1$ becomes specified, the *Euclidean application* still <u>cannot</u> locate line $\overline{AC}$ with respect to the known right triangle BCE

PROBLEM NUMBER 45 *(Ref. Section 18.6)*

GIVEN:

The *Cosine Circle Construction* shown in *Figure 81*

EXPLAIN:

c) Why $\angle \mathrm{CBF} = \theta$, regardless of the value of θ

d) Why $\overline{\mathrm{FG}} = \sin\theta$ when radius $\overline{\mathrm{OA}} = \overline{\mathrm{OB}} = \overline{\mathrm{OC}} = 1$

e) Why $\overline{BH} = \frac{\sqrt{3}}{2} - \frac{1}{2}\tan\theta$ when radius $\overline{\mathrm{OA}} = \overline{\mathrm{OB}} = \overline{\mathrm{OC}} = 1$

SOLUTION:

c) Since the inscribed triangle ABC presented in *Figure 81* is equilateral, each contained angle is equal to 180°/3, or 60°.

All three radii shown in the large circle divide its 360° central angle about the origin (point "O") into three distinct equal angles as follows:

$$Interior\angle AOC = \frac{360^o}{3}$$
$$= 120^o$$

$$Exterior\angle AOC = 360^o - Interior\angle AOC$$
$$= 360^o - 120^o$$
$$= 240^o$$

Now,

$$\angle COD = 270^o - (\theta + 240^o)$$
$$= 30^o - \theta$$

Due to *alternate interior angles* being equal:

$$\angle COD = \angle OCE$$

Or,

$$\angle OCE = 30^o - \theta$$

Since the three radii $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ shown in the large circle are bisectors of triangle ABC, $\angle OCB = 30^o$:

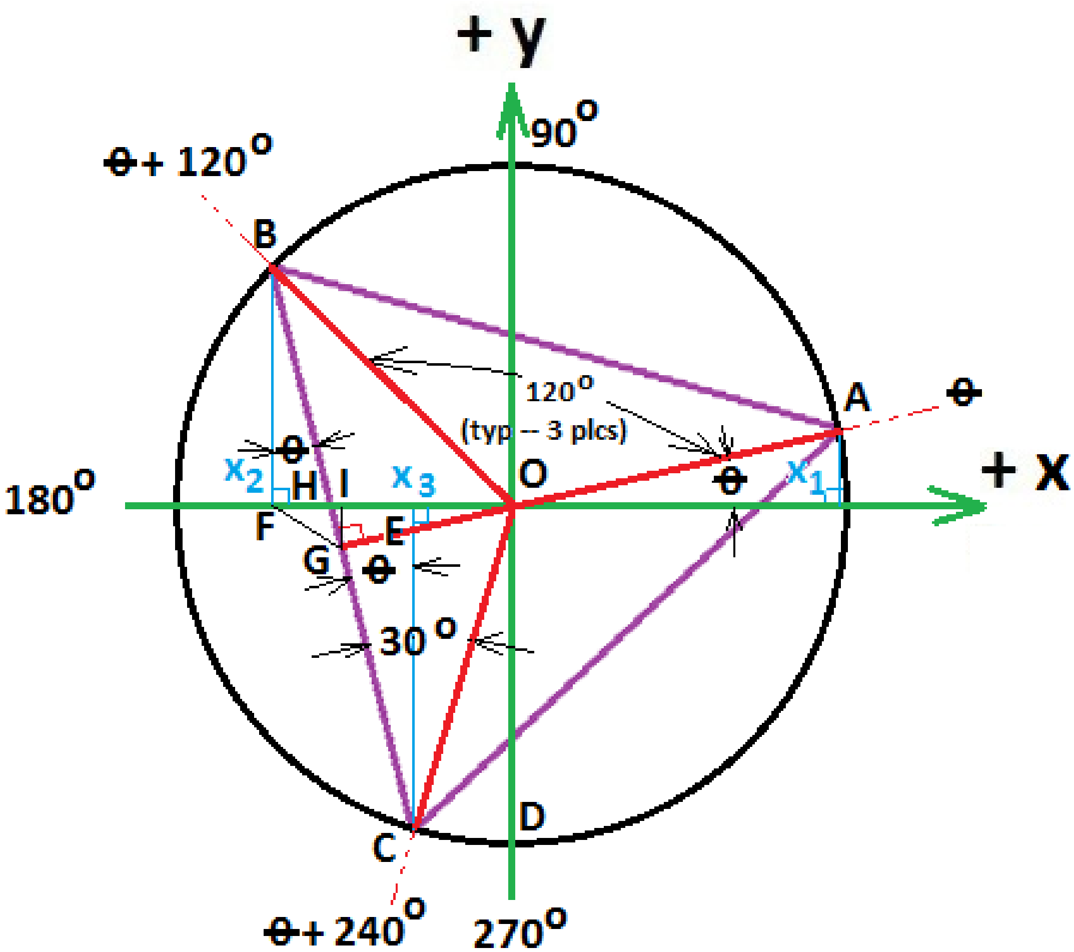
$$\angle ECB = \angle OCB - \angle OCE$$
$$= 30^o - (30^o - \theta)$$
$$= \theta$$

Again, due to the fact that alternate interior angles are equal:

$$\angle CBF = \angle ECB$$
$$= \theta$$

Q.E.D.

Figure 81. ***Cosine Circle*** **Construction (Ref. Figure 47).**



d) For $\overline{OA} = \overline{OB} = \overline{OC} = 1$ in *Figure 81*,

$$\begin{aligned}\overline{OG} &= \overline{OC}\sin 30^o \\ &= (1)(1/2) \\ &= 1/2\end{aligned}$$

Because *opposite angles* are equal:

$$\angle GOF = \theta$$

Furthermore,

$$x_2 = \cos(\theta + 120^o) = -1/2\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$ *[Ref. Section 2.4.1]*

Considering these three above relationships, it becomes evident that:

$$\overline{OI} = ½\cos\theta$$

$$\overline{IF} = \frac{\sqrt{3}}{2}\sin\theta$$

$$\overline{IG} = ½\sin\theta$$

$$\begin{aligned}\tan(\angle FGI) &= \frac{\overline{IF}}{\overline{IG}} \\ &= \frac{\frac{\sqrt{3}}{2}(\sin\theta)}{\frac{1}{2}(\sin\theta)} \\ &= \sqrt{3} \\ &= \tan 60^o\end{aligned}$$

Then,

$$\sin 60^o = \frac{\overline{IF}}{\overline{FG}} = \frac{\sqrt{3}}{2}$$

Or,

$$\overline{FG} = \frac{\overline{IF}}{\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{2}(\sin\theta)}{\frac{\sqrt{3}}{2}} = \sin\theta$$

e) Since $\overline{CF} = \sin(2\theta)$,

$$\overline{BH} = \frac{\overline{BF}}{\cos\theta} = \frac{\sin(60^o - \theta)}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{2} - \frac{1}{2}\tan\theta$$

PROBLEM NUMBER 46 *(Ref. Section 18.6)*

GIVEN:

The *Cosine Circle Construction*

LOCATE:

a) The three *Equation 1* cubic roots within the *Cosine Circle* for $\theta = 20^{\circ}$, and then graphically verify that they reside on the locus of points of its *associated function*, or *curve*

b) The three *Equation 2* cubic roots within the Cosine Circle for $\theta = 20^{\circ}$, and then graphically verify that they reside on the locus of points of its associated function, or curve

SOLUTION:

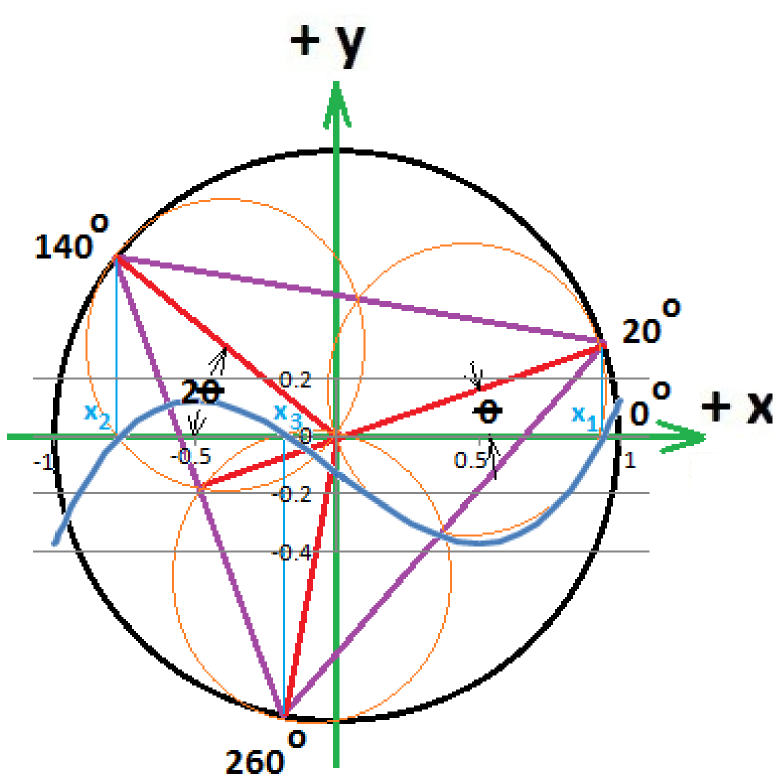
a) *Figure 82* portrays the *Cosine Circle* for $\theta = 20^{\circ}$. It depicts *roots* for *Equation 1* as follows (*Ref. Section 2.4.1*):

$$x_1 = \cos\theta = \cos 20^{\circ}$$

$$x_2 = \cos(\theta + 120^{\circ}) = \cos 140^{\circ} = -\cos 40^{\circ} = -\cos(2\theta)$$

$$x_3 = \cos(\theta + 240^{\circ}) = \cos 260^{\circ} = -\cos 80^{\circ} = -\cos(4\theta)$$

Figure 82. ***Equation 1*** **Cubic Root Locations with the** ***Cosine Circle*****.**



These three *Equation 1 roots* are located at respective *intersection points* of vertical projections emanating from the three vertices of the *Cosine Circle's* equilateral triangle with the x-axis.

They can be located by inscribing three smaller circles inside of the larger circle each of which have their respective centers located half way along the three subtended, generated radii. This enables the respective circumferences of these three smaller circles to reside both upon the origin as well as upon the circumference of the larger circle.

Since,

$$\theta = 20^{o}$$

$$3\theta = 3(20^{o})$$

$$= 60^{o}$$

$$\tau = \frac{1}{2}$$

And,

$$4\cos^3\theta - 3\cos\theta = \cos(3\theta) \qquad \textit{[Ref. Equation 1]}$$

$$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{\cos(3\theta)}{4}$$

$$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{\tau}{4}$$

$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{\tau}{4} = 0$$

$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{0.5}{4} = 0$$

$$\cos^3\theta - \frac{3}{4}\cos\theta - 0.125 = 0$$

Now, the associated function for *Equation 1* is *(Ref. Section 5.1)*:

$$\cos^3\theta - \frac{3}{4}\cos\theta - 0.125 = y$$

This curve is shown superimposed upon *Figure 82* as a plot of *Table 46*. Hence, it is validated that the x_1, x_2, and x_3 roots reside upon this curve.

Table 46. ***Equation 1 Associated Function*** **Calculations.**

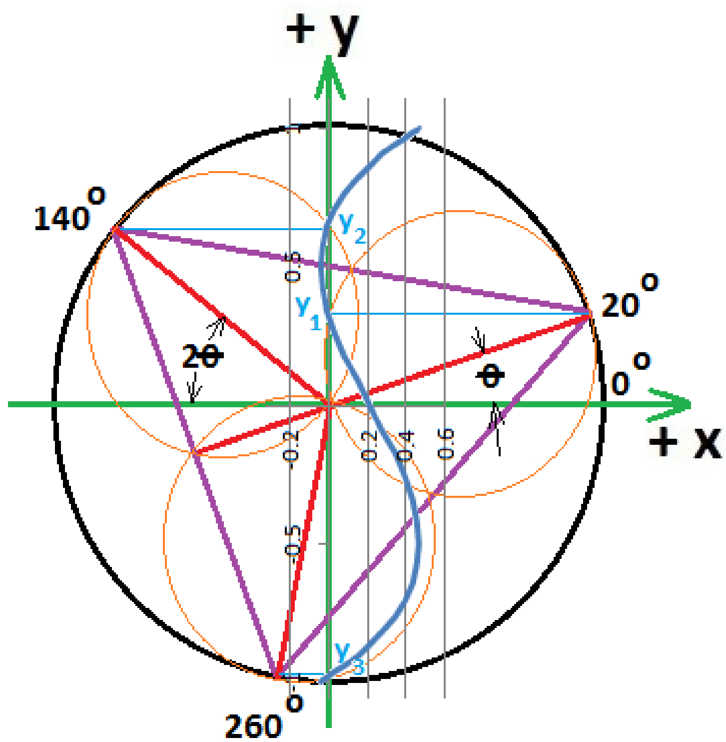
θ Deg	3θ Deg	x = cos θ	cos (3θ) = τ	$\cos^3 \theta$	-3/4(cos θ)	-τ/4	y
0	60	1	0.5	1	-0.75	-0.125	0.125
5	60	0.996194698	0.5	0.98862748	-0.747146024	-0.125	0.116481457
10	60	0.984807753	0.5	0.955112166	-0.738605815	-0.125	0.091506351
15	60	0.965925826	0.5	0.901221065	-0.72444437	-0.125	0.051776695
20	60	0.939692621	0.5	0.829769466	-0.704769466	-0.125	0
25	60	0.906307787	0.5	0.744435602	-0.67973084	-0.125	-0.060295239
30	60	0.866025404	0.5	0.649519053	-0.649519053	-0.125	-0.125
35	60	0.819152044	0.5	0.549659272	-0.614364033	-0.125	-0.189704761
40	60	0.766044443	0.5	0.449533332	-0.574533332	-0.125	-0.25
45	60	0.707106781	0.5	0.353553391	-0.530330086	-0.125	-0.301776695
50	60	0.64278761	0.5	0.265584356	-0.482090707	-0.125	-0.341506351
55	60	0.573576436	0.5	0.188700871	-0.430182327	-0.125	-0.366481457
60	60	0.5	0.5	0.125	-0.375	-0.125	-0.375
65	60	0.422618262	0.5	0.07548224	-0.316963696	-0.125	-0.366481457
70	60	0.342020143	0.5	0.040008757	-0.256515107	-0.125	-0.341506351
75	60	0.258819045	0.5	0.017337589	-0.194114284	-0.125	-0.301776695
80	60	0.173648178	0.5	0.005236133	-0.130236133	-0.125	-0.25
85	60	0.087155743	0.5	0.000662046	-0.065366807	-0.125	-0.189704761
90	60	6.12574E-17	0.5	2.29867E-49	-4.59431E-17	-0.125	-0.125
95	60	-0.087155743	0.5	-0.000662046	0.065366807	-0.125	-0.060295239
100	60	-0.173648178	0.5	-0.005236133	0.130236133	-0.125	0
105	60	-0.258819045	0.5	-0.017337589	0.194114284	-0.125	0.051776695
110	60	-0.342020143	0.5	-0.040008757	0.256515107	-0.125	0.091506351
115	60	-0.422618262	0.5	-0.07548224	0.316963696	-0.125	0.116481457
120	60	-0.5	0.5	-0.125	0.375	-0.125	0.125
125	60	-0.573576436	0.5	-0.188700871	0.430182327	-0.125	0.116481457
130	60	-0.64278761	0.5	-0.265584356	0.482090707	-0.125	0.091506351
135	60	-0.707106781	0.5	-0.353553391	0.530330086	-0.125	0.051776695
140	60	-0.766044443	0.5	-0.449533332	0.574533332	-0.125	0
145	60	-0.819152044	0.5	-0.549659272	0.614364033	-0.125	-0.060295239
150	60	-0.866025404	0.5	-0.649519053	0.649519053	-0.125	-0.125
155	60	-0.906307787	0.5	-0.744435602	0.67973084	-0.125	-0.189704761
160	60	-0.939692621	0.5	-0.829769466	0.704769466	-0.125	-0.25
165	60	-0.965925826	0.5	-0.901221065	0.72444437	-0.125	-0.301776695
170	60	-0.984807753	0.5	-0.955112166	0.738605815	-0.125	-0.341506351
175	60	-0.996194698	0.5	-0.98862748	0.747146024	-0.125	-0.366481457
180	60	-1	0.5	-1	0.75	-0.125	-0.375
185	60	-0.996194698	0.5	-0.98862748	0.747146024	-0.125	-0.366481457

θ Deg	3θ Deg	x = cos θ	cos (3θ) = τ	$\cos^3 \theta$	-3/4(cos θ)	-τ/4	y
190	60	-0.984807753	0.5	-0.955112166	0.738605815	-0.125	-0.341506351
195	60	-0.965925826	0.5	-0.901221065	0.72444437	-0.125	-0.301776695
200	60	-0.939692621	0.5	-0.829769466	0.704769466	-0.125	-0.25
205	60	-0.906307787	0.5	-0.744435602	0.67973084	-0.125	-0.189704761
210	60	-0.866025404	0.5	-0.649519053	0.649519053	-0.125	-0.125
215	60	-0.819152044	0.5	-0.549659272	0.614364033	-0.125	-0.060295239
220	60	-0.766044443	0.5	-0.449533332	0.574533332	-0.125	0
225	60	-0.707106781	0.5	-0.353553391	0.530330086	-0.125	0.051776695
230	60	-0.64278761	0.5	-0.265584356	0.482090707	-0.125	0.091506351
235	60	-0.573576436	0.5	-0.188700871	0.430182327	-0.125	0.116481457
240	60	-0.5	0.5	-0.125	0.375	-0.125	0.125
245	60	-0.422618262	0.5	-0.07548224	0.316963696	-0.125	0.116481457
250	60	-0.342020143	0.5	-0.040008757	0.256515107	-0.125	0.091506351
255	60	-0.258819045	0.5	-0.017337589	0.194114284	-0.125	0.051776695
260	60	-0.173648178	0.5	-0.005236133	0.130236133	-0.125	0
265	60	-0.087155743	0.5	-0.000662046	0.065366807	-0.125	-0.060295239
270	60	-1.83772E-16	0.5	-6.2064E-48	1.37829E-16	-0.125	-0.125
275	60	0.087155743	0.5	0.000662046	-0.065366807	-0.125	-0.189704761
280	60	0.173648178	0.5	0.005236133	-0.130236133	-0.125	-0.25
285	60	0.258819045	0.5	0.017337589	-0.194114284	-0.125	-0.301776695
290	60	0.342020143	0.5	0.040008757	-0.256515107	-0.125	-0.341506351
295	60	0.422618262	0.5	0.07548224	-0.316963696	-0.125	-0.366481457
300	60	0.5	0.5	0.125	-0.375	-0.125	-0.375
305	60	0.573576436	0.5	0.188700871	-0.430182327	-0.125	-0.366481457
310	60	0.64278761	0.5	0.265584356	-0.482090707	-0.125	-0.341506351
315	60	0.707106781	0.5	0.353553391	-0.530330086	-0.125	-0.301776695
320	60	0.766044443	0.5	0.449533332	-0.574533332	-0.125	-0.25
325	60	0.819152044	0.5	0.549659272	-0.614364033	-0.125	-0.189704761
330	60	0.866025404	0.5	0.649519053	-0.649519053	-0.125	-0.125
335	60	0.906307787	0.5	0.744435602	-0.67973084	-0.125	-0.060295239
340	60	0.939692621	0.5	0.829769466	-0.704769466	-0.125	0
345	60	0.965925826	0.5	0.901221065	-0.72444437	-0.125	0.051776695
350	60	0.984807753	0.5	0.955112166	-0.738605815	-0.125	0.091506351
355	60	0.996194698	0.5	0.98862748	-0.747146024	-0.125	0.116481457
360	60	1	0.5	1	-0.75	-0.125	0.125

b) *Figure 83* also portrays the *Cosine Circle* for $\theta = 20^{\circ}$, but depicts roots for *Equation 2* as follows (*Ref. Section 2.4.2*):

$$y_1 = \sin\theta = \sin 20^{\circ}$$
$$y_2 = \sin(\theta + 120^{\circ}) = \sin 140^{\circ} = \sin 40^{\circ} = \sin(2\theta)$$
$$y_3 = \sin(\theta + 240^{\circ}) = \sin 260^{\circ} = -\sin 80^{\circ} = -\sin(4\theta)$$

Figure 83. ***Equation 2*** **Cubic Root Locations with the** ***Cosine Circle*****.**



Such *Equation 2* cubic roots are located at respective *intersection points* of horizontal projections emanating from the three vertices of the *Cosine Circle's* equilateral triangle with the y-axis.

The same three smaller circles constructed in *Figure 82* also serve to locate the *Equation 2 cubic roots* upon the y-axis. Also, for $3\theta = 60^{\circ}$, $\eta = 0.866025403$.

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta \qquad \textit{[Ref. Equation 2]}$$

$$\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\eta}{4} = 0$$

Now, the associated function for *Equation 2* is (*Ref. Section 5.1*):

$$\sin^3\theta - \frac{3}{4}\sin\theta + \frac{\eta}{4} = y$$

This curve is shown superimposed upon *Figure 83* as a plot of the *Table 47.*

Table 47. *Equation 2 Associated Function* Calculations.

θ Deg	3θ Deg	y = sin θ	sin (3θ) = η	$\sin^3 \theta$	-3/4(sin θ)	η/4	x
0	60	0	0.8660254	0	0	0.21651	0.216506351
5	60	0.087155743	0.8660254	0.000662046	-0.065366807	0.21651	0.15180159
10	60	0.173648178	0.8660254	0.005236133	-0.130236133	0.21651	0.091506351
15	60	0.258819045	0.8660254	0.017337589	-0.194114284	0.21651	0.039729656
20	60	0.342020143	0.8660254	0.040008757	-0.256515107	0.21651	0
25	60	0.422618262	0.8660254	0.07548224	-0.316963696	0.21651	-0.024975106
30	60	0.5	0.8660254	0.125	-0.375	0.21651	-0.033493649
35	60	0.573576436	0.8660254	0.188700871	-0.430182327	0.21651	-0.024975106
40	60	0.64278761	0.8660254	0.265584356	-0.482090707	0.21651	0
45	60	0.707106781	0.8660254	0.353553391	-0.530330086	0.21651	0.039729656
50	60	0.766044443	0.8660254	0.449533332	-0.574533332	0.21651	0.091506351
55	60	0.819152044	0.8660254	0.549659272	-0.614364033	0.21651	0.15180159
60	60	0.866025404	0.8660254	0.649519053	-0.649519053	0.21651	0.216506351
65	60	0.906307787	0.8660254	0.744435602	-0.67973084	0.21651	0.281211112
70	60	0.939692621	0.8660254	0.829769466	-0.704769466	0.21651	0.341506351
75	60	0.965925826	0.8660254	0.901221065	-0.72444437	0.21651	0.393283046
80	60	0.984807753	0.8660254	0.955112166	-0.738605815	0.21651	0.433012702
85	60	0.996194698	0.8660254	0.98862748	-0.747146024	0.21651	0.457987808
90	60	1	0.8660254	1	-0.75	0.21651	0.466506351
95	60	0.996194698	0.8660254	0.98862748	-0.747146024	0.21651	0.457987808
100	60	0.984807753	0.8660254	0.955112166	-0.738605815	0.21651	0.433012702
105	60	0.965925826	0.8660254	0.901221065	-0.72444437	0.21651	0.393283046
110	60	0.939692621	0.8660254	0.829769466	-0.704769466	0.21651	0.341506351
115	60	0.906307787	0.8660254	0.744435602	-0.67973084	0.21651	0.281211112
120	60	0.866025404	0.8660254	0.649519053	-0.649519053	0.21651	0.216506351
125	60	0.819152044	0.8660254	0.549659272	-0.614364033	0.21651	0.15180159
130	60	0.766044443	0.8660254	0.449533332	-0.574533332	0.21651	0.091506351
135	60	0.707106781	0.8660254	0.353553391	-0.530330086	0.21651	0.039729656
140	60	0.64278761	0.8660254	0.265584356	-0.482090707	0.21651	0
145	60	0.573576436	0.8660254	0.188700871	-0.430182327	0.21651	-0.024975106
150	60	0.5	0.8660254	0.125	-0.375	0.21651	-0.033493649
155	60	0.422618262	0.8660254	0.07548224	-0.316963696	0.21651	-0.024975106
160	60	0.342020143	0.8660254	0.040008757	-0.256515107	0.21651	0
165	60	0.258819045	0.8660254	0.017337589	-0.194114284	0.21651	0.039729656
170	60	0.173648178	0.8660254	0.005236133	-0.130236133	0.21651	0.091506351
175	60	0.087155743	0.8660254	0.000662046	-0.065366807	0.21651	0.15180159

θ Deg	3θ Deg	y = sin θ	sin (3θ) = η	$\sin^3 \theta$	-3/4(sin θ)	η/4	x
180	60	1.22515E-16	0.8660254	1.83893E-48	-9.18861E-17	0.21651	0.216506351
185	60	-0.087155743	0.8660254	-0.000662046	0.065366807	0.21651	0.281211112
190	60	-0.173648178	0.8660254	-0.005236133	0.130236133	0.21651	0.341506351
195	60	-0.258819045	0.8660254	-0.017337589	0.194114284	0.21651	0.393283046
200	60	-0.342020143	0.8660254	-0.040008757	0.256515107	0.21651	0.433012702
205	60	-0.422618262	0.8660254	-0.07548224	0.316963696	0.21651	0.457987808
210	60	-0.5	0.8660254	-0.125	0.375	0.21651	0.466506351
215	60	-0.573576436	0.8660254	-0.188700871	0.430182327	0.21651	0.457987808
220	60	-0.64278761	0.8660254	-0.265584356	0.482090707	0.21651	0.433012702
225	60	-0.707106781	0.8660254	-0.353553391	0.530330086	0.21651	0.393283046
230	60	-0.766044443	0.8660254	-0.449533332	0.574533332	0.21651	0.341506351
235	60	-0.819152044	0.8660254	-0.549659272	0.614364033	0.21651	0.281211112
240	60	-0.866025404	0.8660254	-0.649519053	0.649519053	0.21651	0.216506351
245	60	-0.906307787	0.8660254	-0.744435602	0.67973084	0.21651	0.15180159
250	60	-0.939692621	0.8660254	-0.829769466	0.704769466	0.21651	0.091506351
255	60	-0.965925826	0.8660254	-0.901221065	0.72444437	0.21651	0.039729656
260	60	-0.984807753	0.8660254	-0.955112166	0.738605815	0.21651	0
265	60	-0.996194698	0.8660254	-0.98862748	0.747146024	0.21651	-0.024975106
270	60	-1	0.8660254	-1	0.75	0.21651	-0.033493649
275	60	-0.996194698	0.8660254	-0.98862748	0.747146024	0.21651	-0.024975106
280	60	-0.984807753	0.8660254	-0.955112166	0.738605815	0.21651	0
285	60	-0.965925826	0.8660254	-0.901221065	0.72444437	0.21651	0.039729656
290	60	-0.939692621	0.8660254	-0.829769466	0.704769466	0.21651	0.091506351
295	60	-0.906307787	0.8660254	-0.744435602	0.67973084	0.21651	0.15180159
300	60	-0.866025404	0.8660254	-0.649519053	0.649519053	0.21651	0.216506351
305	60	-0.819152044	0.8660254	-0.549659272	0.614364033	0.21651	0.281211112
310	60	-0.766044443	0.8660254	-0.449533332	0.574533332	0.21651	0.341506351
315	60	-0.707106781	0.8660254	-0.353553391	0.530330086	0.21651	0.393283046
320	60	-0.64278761	0.8660254	-0.265584356	0.482090707	0.21651	0.433012702
325	60	-0.573576436	0.8660254	-0.188700871	0.430182327	0.21651	0.457987808
330	60	-0.5	0.8660254	-0.125	0.375	0.21651	0.466506351
335	60	-0.422618262	0.8660254	-0.07548224	0.316963696	0.21651	0.457987808
340	60	-0.342020143	0.8660254	-0.040008757	0.256515107	0.21651	0.433012702
345	60	-0.258819045	0.8660254	-0.017337589	0.194114284	0.21651	0.393283046
350	60	-0.173648178	0.8660254	-0.005236133	0.130236133	0.21651	0.341506351
355	60	-0.087155743	0.8660254	-0.000662046	0.065366807	0.21651	0.281211112
360	60	-2.4503E-16	0.8660254	-1.47115E-47	1.83772E-16	0.21651	0.216506351

PROBLEM NUMBER 47 *(Ref. Section 20)*

GIVEN:

The following equation:

$$(3\zeta+\beta)z^2+(3+\gamma)z+(\delta-\zeta)=0 \qquad \textit{[Ref. Section 20]}$$

PROVE:

That the *roots* are always equal to the following, no matter what values of β and γ are applied

$$z_1=\tan\theta$$

$$z_2=-\frac{1}{\tan(2\theta)}$$

SOLUTION:

$$(3\zeta+\beta)z^2+(3+\gamma)z+(\delta-\zeta)=0$$

$$z^2+(\frac{3+\gamma}{3\zeta+\beta})z+(\frac{\delta-\zeta}{3\zeta+\beta})=0$$

Where,

$$(z-z_1)(z-z_2)=0$$

$$z^2-(z_1+z_2)z+z_1z_2=0$$

Comparing like coefficients renders:

$$\frac{3+\gamma}{3\zeta+\beta}=-(z_1+z_2)$$

$$\frac{\delta-\zeta}{3\zeta+\beta}=z_1z_2$$

Since,

$$\zeta=\frac{\delta-\beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta(1-\gamma)+\beta=\delta$$

β ζγ = δ ζ

By substitution,

$$\frac{\beta-\zeta\gamma}{3\zeta+\beta}=z_1z_2$$

Then,

$$3+\gamma=-(z_1+z_2)(3\zeta+\beta)$$

$$\gamma=-(z_1+z_2)(3\zeta+\beta)-3$$

$$-\zeta\gamma=\zeta(z_1+z_2)(3\zeta+\beta)+3\zeta$$

$$\frac{\beta - \zeta\gamma}{3\zeta + \beta} = z_1 z_2$$

$$\beta - \zeta\gamma = z_1 z_2 (3\zeta + \beta)$$

$$\beta + \zeta(z_1 + z_2)(3\zeta + \beta) + 3\zeta = z_1 z_2 (3\zeta + \beta)$$

$$\zeta(z_1 + z_2)(3\zeta + \beta) + (3\zeta + \beta) = z_1 z_2 (3\zeta + \beta)$$

$$\zeta(z_1 + z_2) + 1 = z_1 z_2$$

$$\zeta = \frac{z_1 z_2 - 1}{z_1 + z_2}$$

But,

$$\zeta = \tan(3\theta) = \tan(\theta + 2\theta)$$

$$= \frac{\tan\theta + \tan(2\theta)}{1 - \tan\theta\tan(2\theta)}$$

$$= \frac{(\tan\theta)(-1/\tan(2\theta) - 1}{-1/\tan(2\theta) + \tan\theta}$$

$$= \frac{z_1 z_2 - 1}{z_1 + z_2}$$

Moreover, $z_1 = \tan\theta$ because the *above given equation* was derived from the *3θ Cubic Equation* which expresses such root *(Ref. Section 20)*.

Accordingly,

$$z_2 = -\frac{1}{\tan(2\theta)}$$

Hence, both the z_1 and the z_2 roots determined above apply no matter what values of β and γ are assigned to the *associated Generalized Cubic Equation*.

This explains why such coefficients *drop out* during the calculations conducted above.

PROBLEM NUMBER 48 *(Ref. Section 22.6.4)*

GIVEN:

A circle

DETERMINE:

How the length of pi can be depicted to a significance of *ten decimal places*.

EXPLAIN:

Why *cubic irrational lengths* cannot be mathematically calculated from *rationally-based* ones.

SOLUTION:

Determination:

An arbitrarily applied, or given *rational length* easily can be represented as the multiple of two *cubic irrational*, or even *transcendental lengths*.

For example:

$$\frac{13}{9} = \frac{13}{9}\left(\frac{\tan 20^o}{\tan 20^o}\right)$$

$$= \frac{13\tan 20^o}{9\tan 20^o}$$

$$= \frac{4.7316130455}{3.2757321084}$$

Now, the *transcendental length* pi may be multiplied by the transcendental length sin 80° in order to produce another transcendental *length* as follows:

$$\pi \sin 80^o = 3.093864802...$$

$$\pi(0.9848077530...) = 4(0.77346620052...)$$

Moreover, all values, except for π in such above equation furthermore very closely could be approximated as actual rational numbers, down to a significance of ten and eleven decimal points, respectively; being well beyond the accuracy of what the naked eye could detect.

Such estimated result is furnished directly below, whereby all constructible rational numbers thereby can be expressed algebraically as follows:

$$\pi\left(\frac{984,807,753}{1,000,000,000}\right) = 4\left(\frac{77,346,620,052}{100,000,000,000}\right)$$

$$\pi\left(\frac{984,807,753}{1,000,000,000}\right) = 4\left(\frac{19,336,655,013}{25,000,000,000}\right)$$

$$\pi(L) = 4(T)\text{ ; or}$$
$$\pi L = 4T$$

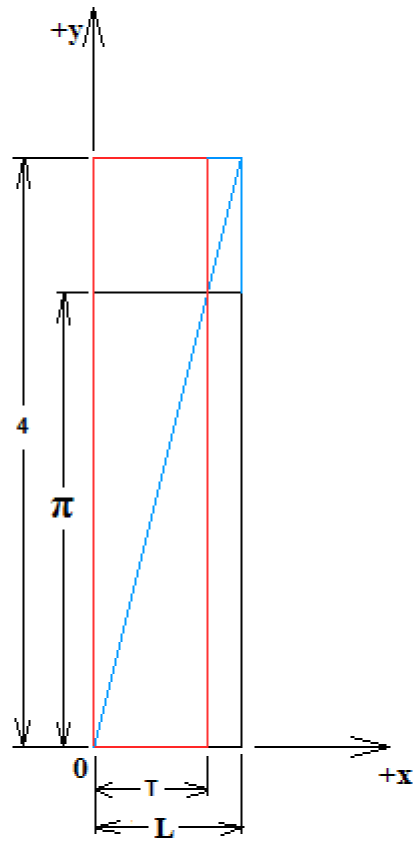
Consider: Such above described *rational lengths* of 4, T, and L could be ***geometrically constructed*** from an arbitrarily applied, or given length of unity *(Ref. Section 9.1 Conclusion)*.

In this example, there is little need to attempt to reduce the rational length, T, any further than is indicated. This is because it is necessary only to know that a *rational length* $T = 19{,}336{,}655{,}013/25{,}000{,}000{,}000$ can be made use of to ***geometrically construct*** another length that very closely approximates the actual value of pi.

From the equation $\pi L = 4T$, as determined above, the proportion $\frac{\pi}{T} = \frac{4}{L}$ readily can be established.

Whereby, a very close estimation of the *length pi* thereby can be identified from the ***geometric construction*** of two similar right triangles whose sides respectively consist of drawn *rational lengths* 4, T, and L *(Ref. Figure 84)*; thereby becoming located at the juncture of the common hypotenuse of such two similar right triangles and the vertical line ascribed at x = T.

Figure 84. Determination of Pi.



To *ten decimal places*, the ***geometrically constructed*** length pi then would be equal to:

$$\pi = 3.1415926536$$

Explanation:

Since *transcendental lengths* describe *decimal sequences* which are considered to *continue on indefinitely*, they cannot be replicated by a long-hand division, indicative of *rational numbers*, whose quotients begin to repeat themselves.